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Physics of Selective Systems: Computation and Biology

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The statistical thermodynamics of systems displaying selective behavior is used to discuss some important ultimate physical limitations of computers and biological systems. These cluster around communication of information, measurement, and irreversible processes. The most fundamental limitation is irreducible increase of entropy accompanying selective acts like measurement or preparation. Relevant theory of machines (automata, Turing machines) and issues involved in physical realizations of those machines are discussed. Quantum measurement, the Einstein–Podolsky–Rosen paradox, the fundamental importance of irreversibility, information and entropy, and their relation to Goedel's theorems on completeness and consistency of formal systems are analyzed. Irreversibility of measurement appears necessary to provide quantum mechanics with the incompleteness needed to avoid inconsistency. Motivation and justification of computer paradigms for fundamental modeling of biological systems is given.

1. INTRODUCTION

Computers and living systems share many characteristics: they behave selectively, show tremendous differences in response to similar inputs, and are nonequilibrium systems, generally metastable. They usually require ongoing dissipation to maintain their characteristic behaviors, which depend on stored information. Both involve loosely coupled subsystems, use elaborate internal communication and control systems, and key interactions within them and with their environments are typically irreversible and "all or nothing" (nonlinear). This paper probes some ultimate limitations on complex system function connected with irreversibility.

Selective behavior presupposes a system exhibiting that behavior by interacting with another system (environment, world) with respect to which that behavior is displayed. The boundary between them need not be fixed or sharp; any selective system (SS) can be viewed as a subsystem of a global SS. The SS usually consists of many interacting subsystems which can be SSs individually; a subset of the set of subsystems is then an (internal) environment of any one. *Discussion of ultimate limitations in SS performance has all the problems of quantum measurement and preparation to confront as its simplest case.*

As an SS is more macroscopic than microscopic, we use the following idealization to deal with it. We endow it with some (generally large) set of discrete gross states (B-states) which are *behaviorally* distinguishable, i.e., there exist well-defined input sequences for which distinguishable (i.e., measurable) output behaviors are obtained for any pair of states. They can have the same or very different energies, and are generally mutually inaccessible ergodically unless an appropriate sequence of B-state transitions is induced by receipt of an input (control) sequence. Output sequences (behavior) are functions of input and B-state sequences. B-state change can alter the *boundary conditions* defining the solutions of dynamical equations describing an SS and thus, generally indicates preparation of the SS anew. Both the phase space and the quantum language needed to describe the system can change, as can gross operative constraints or internal boundary conditions, as with changes in setting a switch, or turning a valve. Different behaviors can be programmed, via controlled impositions and relaxations of boundary conditions or constraints, and changing a program can even be viewed as preparing a new SS.

The centrality of boundary conditions, with particular reference to biology, is stressed in Rothstein (1979a). The present paper is a sequel to this one (which contains many references); both are sequels to Rothstein (1971). This last reviews many aspects of the informational generalization of physical entropy, and notes the immediate connection between it, specifications and boundary or initial conditions. Also relevant is Rothstein (1967). Operational implementation of a specification or boundary condition, preparing a system, or measuring a system, are macroscopic irreversible procedures. We find it natural, therefore, to view measurement as logically prior to quantum mechanics in the sense that *the very possibility of confronting quantum mechanics (QM) with experience presupposes a macroscopic concept of measurement, and preparing a quantum system presupposes a macroscopic preparation procedure.*

The discussion of mixtures, measurement, and irreversibility of von Neumann (1955) is consistent with this operational approach, and clearly exhibits the arbitrary placement of the division between system observed or prepared and means used to observe or prepare it. Bohr's (1935) emphasis on the quality of wholeness of QM systems again points to the logical jump necessary to go from quantum object to macroscopic objects. Ordinarily it is effectively absorbed into a boundary condition. The different conditions on the two "sides" are handled by different concepts and methods, the boundary itself being both intractable and arbitrary. Until system of interest and means of observation or preparation interact, or after their interaction, when they are again separate, both entities can be treated individually as quantum systems. As they are effectively a single system while interacting, the transition to two systems (reduction of the wave function, EPR thought experiment) is a *jump in viewpoint* or *specification* not described by the Schrödinger equation of the compound system *which is reflected in the irreversible nature of measurement. This irreversibility is posited, not derived*, and we believe *this is necessarily the case.*

Boundary conditions determine what specific solutions of the dynamical equations are appropriate. Controlled (prepared) by the experimenter, they are part of the *specification* of the physical problem, and therefore are *not derivable from the dynamical equations of the system.* To maintain the opposite requires one to swallow a self-referential fallacy or an infinite regression. The latter corresponds operationally to the fact that regarding the compound system as a quantum system requires yet another system to prepare it in a state ψ_{12} which will evolve into two subsystems in states ψ_1 and ψ_2.

In any SS (which can be an SS and its world), two subsystems must interact to make information transfer possible between them. One can say either that the information source prepares the receiver, changing its B-state from one in which the message has not been received to another indicating it has, or that the receiver is a means for observing the source, receipt of the message indicating the source has entered a postemission B-state. But in a computer, say, *interest generally centers in sequences of changes of B-state.* As these are boundary condition changes, measurement or preparation acts which are essentially irreversible, computer "metabolism" (or biological metabolism) is a dissipative process *unless there are no B-state changes.*

In principle there is no reason to believe it impossible to construct systems, biological or computational, in which the only irreversible steps are preparation of the system initially, and final measurement of its time-evolved state. Viruslike behavior, where a viral system is involved in irreversible activities essentially only during its reproductive and measurement (i.e., attack or host cell recognition) phases is suggested as a biological case of this kind. The corresponding computational concept is reversible computing, so named because time evolution of the system between programming (preparation) and final read-out of the result is governed entirely by its dynamical equations, and thus reversible. We stress that neglect of irreversibility associated with measurement or preparation of initial or boundary conditions implies violation of the second law (Szilard, 1929) and the paradoxes of Loschmidt and Zermelo (Rothstein, 1957, 1974). The work of Bennett (1973, 1982), Benioff (1980, 1982), Fredkin and Toffoli (1982), Landauer (1961), Likharev (1982), and Toffoli (1981) shows that on classical, quantum, and thermodynamic grounds general computations can be done reversibly in the above sense, where entropy production is "exported" into the boundary conditions, and/or the computation is done very slowly. Computing at realistic rates, or having a *general purpose computer* (as contrasted to a special purpose or fixed program computer) may entail prohibitive difficulties for entropy export into a fixed boundary condition. Conditional branch instructions, for example, seem to require both a measurement to determine whether or not the condition holds, and the setting up of a new boundary condition depending on the outcome, unless both paths have been "built in." The latter appears to require that a general purpose machine with several branch instructions be exponentially infinite. A finite machine would have to acquire one bit of information at each branch, and by Szilard (1929) and Rothstein (1951) this entails an entropy cost not less than $k \ln 2$. Whether as an initial setup cost, information acquisition costs, or information read-out cost, *there is an irreducible entropy cost for a computation, independent of temperature.*

For surveys of other fundamental limits of general interest here see Keyes (1981) (digital processing), Wyner (1981) (information theory), Personick (1981) (optical communication), Kogelnik (1981) (integrated optics), and Cooper (1981) (field-effect devices). Among other sources of system limitations we note the following. Programming *languages* or *algorithms* can be inefficient. The monitor, or operating system, may not make optimal use of computer resources. Concurrent processes may be in competition for the same resource. Queues can develop in one area with other areas underutilized, and routines to relieve the situation may even intensify the problem. It is not clear what fundamental physical limitations might exist here.

2. SELECTIVE BEHAVIOR, *B*-STATES, AND AUTOMATA

We need to define *B*-states in a manner achieving realistic reconciliation of the notions of state used in computer science and in physical science. The first is discrete, the second frequently continuous. Time variation of the first is freely programmable; equations of motion of physical systems are unalterable dynamical laws. The classes of computations performable by nondeterministic finite state automata or Turing machines are respectively identical to those performable by deterministic finite state machines or

Turing machines. In physics the distinction is a fundamental difference between classical and quantum mechanics which played a conceptually revolutionary role. In computers, there is no problem in principle in specifying the state and subsequent behavior of the system completely. In physics, quantum mechanics sets limits on both. In computer science, given the state of a deterministic system, it is generally *impossible to specify the preceding state*; physics is dynamically reversible. In a sense clear from the above, computer science is irreversible and deterministic, physics reversible and nondeterministic.

Clearly, neither pure quantum states nor classical states can generally be identified with B-states. We can represent B-states as mixtures or generalizations of the concept of thermodynamic state. Because they are constructs in operationally different languages, those of microscopic and macroscopic states, respectively, we should distinguish between them.

The simplest state concept in computer science occurs with the finite state automaton (FSA), and can be introduced simultaneously with inputs, outputs, and transitions (see Hopcroft and Ullman, 1979; Denning, Dennis, and Qualitz 1978; Ginzburg, 1968). Write

$$A=(Q,\Sigma,\Omega,M,N) \tag{1}$$

where A is an FSA, presented as a 5-tuple. Here

$$Q=\{q_0,q_1,\ldots,q_{m-1}\} \tag{2}$$

is a finite set of states,

$$\Sigma=\{\sigma_0,\sigma_1,\ldots,\sigma_{n-1}\} \tag{3}$$

is a finite set of inputs (input alphabet), and

$$\Omega=\{\omega_0,\omega_1,\ldots,\omega_{p-1}\} \tag{4}$$

is a finite set of outputs (output alphabet). M and N are mappings, whose common domain is the Cartesian product $Q\times\Sigma$ with respective ranges Q and Ω,

$$M:Q\times\Sigma\to Q \tag{5a}$$

$$N:Q\times\Sigma\to\Omega \tag{5b}$$

called, respectively, the next state function and the output function. Nondeterministic machines have M or N are weakened to be relations rather than functions. In many cases, e.g., formal language recognition, it is necessary to designate a standard initial state q_0, often included as a sixth symbol in (1). Output symbols are associated either with states (Moore machine) or with transitions (Mealy machine), the latter useful in communication circuits, the former in systems capable of storing a finite amount of information. In a language recognizer (acceptor) it is customary to suppress Ω and N in (1), include q_0, and designate a subset F of Q as "final" or accepting states. A word (or string) w consisting of symbols of Σ is accepted if and only if, when A is started in q_0, the sequence of state changes induced by feeding in w, symbol by symbol, terminates in a state q_f of F. The output Ω is then "accept" for $\{q_i \in F\}$ and "reject" for $\{q_i \in Q - F\}$. Instead of (1) we have

$$A = (Q, \Sigma, M, q_0, F) \tag{6}$$

It is often convenient to present A as a labeled directed graph. Its nodes correspond to states, generally drawn as circles containing the labels q_i, which for Moore machines also contain the appropriate output $\omega_j(q_i|\omega_j$ is a frequently used notation). For acceptors the ω_j label is replaced by a double concentric circle for all q_i in F. A short arrow with head on the circle indicates q_0. If input σ_i induces a transition from q_j to q_k, an arrow is drawn from q_j to q_k and labeled by σ_i; in a Mealy machine the output, say ω_l, also labels the arrow, with $\sigma_i|\omega_l$ a commonly used notation.

Figure 1 shows a simple 3-state machine; when fed an arbitrary string w over the binary alphabet

$$\Sigma = \{0, 1\} \tag{7}$$

it computes the remainder on division by 3 if the string is interpreted as a base 2 integer. Were one only interested in multiples of 3, state [0] would have a double circle. The outputs 0, 1, 2 are used as state names; output i comes only from state q_i so q can be suppressed. The square brackets emphasize that entire remainder classes consist of equivalence classes of strings, interpreted as transformations on the set of states. Using the postoperator notation

$$q_i w = q_j \tag{8}$$

to mean that when A is in state q_i, if word w is fed in, symbol by symbol, the new state will be q_j, and letting E mean "is equivalent to," we have

$$w_1 E w_2 \quad \text{if} \quad \text{and only if } q_i w_1 = q_i w_2, \quad q_i \in Q \tag{9}$$

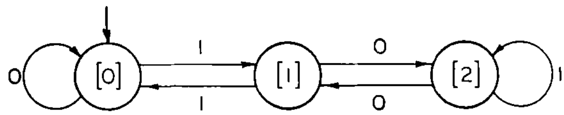

Fig. 1. FSA which classifies binary strings, interpreted as integers, according to the remainder on division by 3. It is a group machine whose group is D_3.

Labels designating blocks (equivalence classes) of equivalence relation E can serve as state names. In general any equivalence relation of finite index (i.e., with a finite number of blocks) satisfying (9) defines the minimal state automaton capable of classifying all strings over Σ according to block membership. Strings over Σ form a free semigroup (S, say) under concatenation with Σ as generators. As there is an identity (the null string λ) the semigroup is a monoid by definition. We can now see that there is a finite semigroup of transformations, S_A, on the states Q of A, induced by elements of S, which is a homomorphic image of S. The homomorphism involved is a mapping

$$h: S \to S_A, S_A = S/E \tag{10}$$

i.e., S_A is the quotient monoid of S by equivalence relation E. In the example of Figure 1, S_A is a group isomorphic to D_3 (symmetry group of the equilateral triangle). Recall that a group is a monoid with an element inverse to any given element, i.e., for any $x \in G$ there exists a unique $x^{-1} \in G$ such that $xx^{-1} = e$, where e is the identity. In automata theory, A is a *group machine* (i.e., S_A is a group) if and only if for every $\sigma_i \in \Sigma$ we can find a string α_i over Σ such that $\sigma_i \alpha_i E \lambda$. Alternatively, S_A is a group if and only if every $\sigma_i \in \Sigma$ permutes the states. A σ_i that transforms any state into one particular state is called a reset input, and a machine which is reset by all inputs is called a *reset machine*. An example is shown in Figure 2, which gives the remainder, on division by 2, of any binary number. The monoid of a reset machine is extremely simple: $\sigma_i \sigma_j E \sigma_j$. Resets are idempotent, like projection operators $P(P^2 = P)$.

Two machines are equivalent if they always produce the same outputs for the same inputs, i.e., have the same input–output behavior. An important structural theorem states that the general FSA is equivalent to cascaded or direct products (essentially like series–parallel connections) of simple machines of two types, group machines and two-state reset machines.

We now try to combine physical and automaton state concepts into a generalization including both. We consider both what is involved in physical realization of an abstract automaton, and how a physical system can be presented as an automaton.

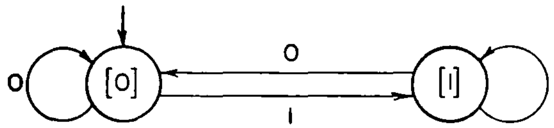

Fig. 2. FSA which classifies binary strings, interpreted as integers, according to the remainder on division by 2. It is a reset machine.

To realize a particular automaton A requires an information source to emit Σ-signals and a physical system with a set of gross states in one-to-one correspondence with Q, in any of which, on receipt of a Σ-signal, the system either emits an Ω-signal and makes a transition to the same or another gross state (Mealy machine), or which, after making the transition, has the Ω-signal available as the result of a measurement made on the system in the new gross state which does not perturb the gross state (Moore machine). These gross states partition the phase space of all relevant microstates of the system. Until receipt of a Σ-input ergodic wandering of its representative point keeps it within a region of phase space corresponding to gross state B_1. Receipt of a Σ-input induces a transition to gross state B_2. As anticipated by the notation we identify the blocks (ergodic components) of this partition with B-states. Whether a B-state is taken as an equilibrium, steady, or nonsteady macroscopic state is of no importance here. The buffetings of the representative point can be random shocks as long as they do not cause B-state changes with appreciable probability; one measure of system unreliability is the probability that a B-state change is random, rather than controlled. Low-temperature operation or making the barrier E against spontaneous B-state transitions very large ($E/kT \gg 1$) enhances SS reliability.

There is much similarity here to the operational discussion of quantum measurement by Lamb (1969). He considers a three-stage process to be involved: initial state preparation, perturbation, and examination for the probability that a particular final state is occupied. The first stage, for a particle, say, might involve catching it in a potential well $U_1(x)$ chosen to make its lowest energy eigenfunction be the state $\psi(x,0)$ which one desires to prepare. Let its excess energy radiate away and the state is prepared. In the perturbation stage, we suddenly turn off U_1 and turn on the perturbation V, after which the wave function evolves according to Schrödinger's time-dependent equation. Then we turn off V and turn on another potential cleverly chosen to yield the occupation probabilities of the states it is desired to measure. The underlying similarity between this and our discussion is easy to see—the first well corresponds to initial B_1, the second to final B_2, the perturbation to channel. Our simple theory gives the additional insight that lower barriers suffice at low T.

We require either that the Σ-input surmount E (impractical), or that it function catalytically to allow a phase space channel to develop between initial and final Q-blocks. In the latter case the probability that the representative point wanders quickly into the final block must be overwhelming, and after it has done so, the channel must be closed again (unless it is supposed to return to the initial B-state). The first step requires entropy increase, the second needs energy, which is usually dissipated. The justification for this was given originally in Rothstein (1979) in an analogous biological context.

Let ϕ_1 and ϕ_2 be the phase volumes of blocks corresponding to B-states B_1 (initial) and B_2 (final), respectively. Their entropies will then be

$$S_1 = k \log \phi_1 \tag{11}$$

$$S_2 = k \log \phi_2 \tag{12}$$

When the channel is opened the accessible phase volume ϕ becomes

$$\phi = \phi_1 + \phi_2 \tag{13}$$

The corresponding entropy S is

$$S = k \log(\phi_1 + \phi_2) \tag{14}$$

The entropy *increase* consequent to this increase in phase volume is given by

$$\Delta_1 S = S - S_1 = k \log(1 + \phi_2/\phi_1) > 0 \tag{15}$$

When the channel is closed, trapping the representative point in ϕ_2, the entropy had to be *decreased* by an amount

$$\Delta_2 S = S - S_2 = k \log(1 + \phi_1/\phi_2) > 0 \tag{16}$$

By the second law of thermodynamics an entropy increase occurs elsewhere of at least this much. Ordinarily an amount of energy ΔE would have to dissipated to reduce the entropy by $\Delta_2 S$. Taking T as the temperature at which ΔE is dissipated,

$$\Delta E \geqslant T\Delta_2 S = kT \ \log(1 + \phi_1/\phi_2) > 0 \tag{17}$$

Clearly, by taking T low enough, ΔE can be made arbitrarily small, but even with approach to absolute zero the total entropy change ΔS can not be reduced below the value given by (15). It is thus a *fundamental limitation* (as noted earlier) required by thermodynamics, which is essentially irreversibility of measurement or preparation.

The idealization is often made in thought experiments that opening or closing a channel is performed with arbitrarily small energy or entropy cost. We believe this is admissible here; channel opening can then be done by a Σ-signal. In effect the energy and entropy cost, if any, can be charged to the Σ-source preparation as in reversible computing. But closing the channel must occur sufficiently long after opening to permit the representative point to stream into ϕ_2 (or $\phi_2 + \phi'$ with subsequent diminution by ϕ') but not so long after that it might have streamed back into ϕ_1. The larger ϕ_2/ϕ_1 (or $\phi_2 - \phi_1$), the less onerous is this "Poincaré recurrence" condition. Suppose

$$\phi_2/\phi_1 = n \gg 1 \tag{18}$$

Then

$$\Delta S \approx \Delta_1 S \approx k \log n \tag{19}$$

Reliable closure may thus require large (energy or) entropy cost in practice. This can be done with amplification: the Σ-input is the trigger or trap-door opener; the energy released (and usually dissipated) closes it.

To conclude this section we verify that any physical system can be viewed as an automaton and how resets correspond in a natural way to irreversible interventions in the dynamical evolution of a system.

Consider a system whose state is determined by n variables $x_1, x_2, \ldots, x_n$, subject to

$$dx_i/dt = X_i(x_1, \ldots, x_n) \tag{20}$$

The state $\varphi(x_1 \ldots x_n)$, which depends on t through the x_i, is "told" to change to state $\varphi + d\varphi$ in time dt, where

$$\begin{aligned} d\varphi &= \Sigma(\partial\varphi/\partial x_i)(dx_i/dt)\,dt \\ &= (X \cdot \nabla\varphi)\,dt \end{aligned} \tag{21}$$

where the vector X is given by

$$X = (X_1, X_2, \ldots, X_n) \tag{22}$$

We can say that state φ, with input X, makes a transition to state $\varphi + d\varphi$. The output set is $x_i(t+dt)$, where

$$x_i(t+dt) = x_i(t) + X_i\,dt \tag{23}$$

Alternatively, the n-vector

$$x = (x_1, x_2, \ldots, x_n) \tag{24}$$

can be called both state and output. In QM φ is the wave function and (20) the time-dependent Schrödinger equation

$$H\phi = -(h/2\pi i)\,\partial\varphi/\partial t \tag{25}$$

From Lie group theory the input monoid of (20) or (25) is a realization of the one-parameter translation group (n-dimensional or infinite-dimensional flows, respectively), namely, the group of contact transformations representing time translation of the system. Reversibility corresponds to existence of inverses. Write the solution of (20) as

$$x(t) = T(t)x(0) \tag{26}$$

Here $x(0)$ is the value of x [defined by (24)] at $t=0$, $x(t)$ its value at time t obtained by integrating (20), with $\{T(t)\}$ the corresponding group of time-displacement operators on x. $T(0)$ is the identity, and using concatenation to express the group operation, we have

$$T(t_1)T(t_2) = T(t_1+t_2) = T(t_2)T(t_1) \tag{27}$$

with $T(-t)$ the inverse of $T(t)$. In QM we have

$$T(t) = \exp(-iHt/\hbar) \tag{28}$$

the unitary group in Hilbert space for Hermitean H, and ϕ places x in (26).

Reset transformations, being many–one, have no inverses. *It is thus impossible for a reset transformation to belong to the set $T(t)$ either classically or quantally.* A physical reset transformation can therefore only be an external intervention in the history of a system, i.e., a preparation act, a setting up of a new initial condition. Conversely, a procedure which takes a system in *any* of some class of states and prepares it in some *particular* state is a reset procedure and irreversible. This generalizes the operational discussion of Rothstein (1957, 1974) and shows how *the time-evolution group* *generalizes to a time history monoid when external interventions occur. Group character can only be restored by redefining the system to include the intervention means thereby exporting the irreversibility into a new initial condition* (setup charge), exactly as in the discussion of Loschmidt's paradox in the references just cited.

3. TURING MACHINES, UNDECIDABLE QUESTIONS, AND BIOLOGY

The FSA has limited information storage capacity. The most general model of computation thought possible is the Turing machine (TM). It consists of an unbounded tape memory and an FSA which can access it. For convenience the tape is ruled into squares (side equal to tape width) which can contain any symbol of tape alphabet Γ or a "blank." All inputs are read from, and all outputs are printed on the tape, by the FSA (also called finite state control or simply control). Indexing the squares by the positive integers, and calling the square currently scanned by the FSA its address, permits the following definition of a "move" of the TM. In (current) state q_i, scanning symbol (input) s_j, the machine prints (output) s_k, and moves left or right to an adjacent square in (next) state q_l. Behavior of the machine for all time is determined by the table of possible moves and the initial configuration (q_0, α, i). Here q_0 is the initial state of the FSA, α is the initial string over Γ printed on the tape, and i is the initial address. The FSA also usually has a set of final states, F; when one is entered the computation *halts*. The table of moves can be given as a list of quintuples or as a so-called functional matrix; either expresses the mapping

$$T: Q \times \Gamma \to Q \times \Gamma \times M \tag{29}$$

Here

$$M = \{L, R\} \tag{30}$$

meaning that the FSA moves either left (L) or right (R). Two-way movement is essential; restriction to one-way movement makes the TM equivalent to an FSA. No additional generality is obtained by augmenting set M by S ("stationary"), corresponding to no change in address, or by using two-way infinite or multiple tapes (including infinite multiplicity, say cells in an unbounded M-dimensional space). No additional generality results if the finite control is nondeterministic, if the squares can hold strings of symbols rather than a single symbol, or if the finite control can make big jumps to the next address (including vector jumps in high dimensionality tape space). No additional generality results from the use of multiple automata, up to and including having an automaton stationed at every square, with communication means provided between them. These and other clever devices make tremendous difference in the *speed* with which computations can be carried out. The modern high-speed general purpose computer is a universal TM (defined below) sped up by such means.

The universal TM, U say, is one which can simulate an arbitrary TM, T. States, symbols, and moves of T are encoded as strings over the alphabet of U, and the encoded functional matrix (list of quintuples) of T included, along with the encoded initial configuration of T, on the tape of U. The states, symbols, and moves of U are designed to enable U to carry out a universal simulation algorithm, whereby it processes the string describing the initial configuration of T, taking it through encoded versions of the configurations through which T would pass, and halting when a configuration corresponding to one at which T halts is reached. Proof of existence of U is constructive. One can regard U as a general purpose machine which carries out the *program* to which T corresponds. Arbitrary T can be regarded either as a specific (special purpose) machine or as the algorithm or procedure it executes. We can view U as a specific machine or as an ensemble of potential machines (incomplete machine) which turns into an actual machine when provided with a program.

Turing (1936, 1937) invented both T and U to shed light on problems in the foundations of mathematics, particularly decision problems. Goedel (1931, 1934) had shown that in any formal system M broad enough to include arithmetic, (a) one can exhibit arithmetic propositions of rather elementary nature which are evidently true yet cannot be deduced within M (i.e., if a deduction existed it would imply a contradiction), and (b) the consistency of M cannot be deduced in M (i.e., if M could be proved consistent by means contained entirely within it, it would be inconsistent). As the class of systems like M includes systems like the predicate calculus and calculus of sets, there was a tremendous intellectual upheaval and many attempts to get around these theorems on the completeness and consistency of formal systems (a complete system is one in which every theorem *true in the system* is *provable within the system*, a consistent one never leads to proof of a contradiction). As discussion often bogged down in cloudy questions of semantics, Turing sought to play the formal system game in a way so formalized that even a machine could do it. He simulated the way a logician or mathematician would test the validity of a proof within a given formal system, using a finite set of rules for manipulating symbols and a conceptual machine which followed those rules. Goedel's theorems not only were not evaded but assumed particularly illuminating forms, the most famous being the algorithmic unsolvability of the halting problem for Turing machines; the problem is as follows. Given an arbitrary TM, T, and an arbitrary string α for it to process, will T ever stop computing, say by entering a state $q_f \in F$? To answer the question we need a general procedure, i.e., a TM, which takes a properly encoded version of T's table and α, and computes "yes" or "no." Turing showed such a TM does not exist.

Many undecidable questions (equivalently, noncomputable numbers) are now known. A frequently used proof of undecidability for a particular question is to show that existence of a decision method implies solving the halting problem. For example, can one decide whether, during the course of a (general) TM computation, a particular configuration will actually occur, i.e., is the latter accessible from the initial configuration? The answer is no (take occurrence of the particular configuration as the halt condition of an essentially identical associated TM). This has been used to argue the unpredictability of evolution (and history or psychology for that matter) even in a "perfect" theoretical biophysics by Rothstein (1979).

Discussion of physical Turing machines introduces no physics not already discussed in Section 2; the only novelty is the infinite tape. It is both external information source for the FSA and object prepared which carries its stored output. It has no upper bound to its information storage capacity, but at any time only a finite amount of tape will have been in interaction with the control. The infinite tape ("buy more tape if you really need it") is an idealization no less justified than $t \to \infty$ ("wait longer if it makes a real difference") for coming to equilibrium or $x \to \infty$ ("go out further if it makes a real difference") for a beam scattering problem, *ordinarily*. Exceptions could arise for computations so complex that the tape needed would more than exhaust all the matter in the known universe, say. While such computations can easily be devised, their ad hoc and bizarre nature is *not yet* likely to trouble theorists in physics or biology. It thus still seems reasonable to expect future scientific theory, *when formalized*, to remain calculable in principle on a Turing machine, and therefore guaranteed to contain undecidable questions, to be *incomplete*, and if completeness be *forced*, to become *inconsistent*. It is true of present theories, including quantum mechanics; the connection between this and EPR is considered in Section 4.

In science there are famous impediments to complete knowledge about the universe posed by relativity (arising from the finite velocity of light) and quantum mechanics (uncertainty principle, complementarity). Thermodynamics also sets limits on how well the state of the universe, or any part of it, can be known as a result of measurement. This results in the existence of physically undecidable questions grounded in the first and (particularly) the second law of thermodynamics. These have been investigated in Rothstein (1959, 1964). We have long felt that in the physics of the future all undecidable questions ought to be derivable from the Goedel–Turing species of undecidability (see the comment on Reference 10 in Rothstein, (1967), and sensed that the key to showing it might emerge from mutual confrontation of all the "undecidabilities." We believe progress has been made along this line in Section 4, but must first define "undecidable" *operationally.*

Any physical system can be viewed as a (special purpose) computer programmed to simulate that system. *Anything observable in the operational sense must be computable (predictable) within the formal system constituting the theory of that physical system if the theory is complete.* We can always read the tape output (measurement of the time-evolved prepared system) computed from the operationally defined tape input (specification of the operationally prepared state of the system). If, therefore, two equally competent, non-mutually-interfering observations of an evolved state contradict each other, either the underlying theory is inconsistent, or the observed situation is one where the theory cannot decide between them. With respect to such a decision the theory is therefore *incomplete*; if it were not it would be *inconsistent.* This is *not* a defect of the theory, for it is inherent in *all* formal systems. We argue in Section 4 that *irreversibility of measurement is what preserves quantum formalism from contradiction.* We believe this supports the idea that the second law of thermodynamics is like a Goedelian theorem for the underlying fundamental dynamics.

We close this section with a few remarks on theoretical biology (most also apply to psychology and social science), where one might expect to encounter many undecidable questions. The axiomatic–deductive nature of mathematics has been formalized by modern theories of computability in such a way that few doubt the essential equivalence of formal theories and Turing machines. Modern general purpose computers are Turing machines capable of calling vast numbers of computationally efficient routines. This contrasts sharply to logical "atom-by-atom" TM operation. Nowadays, one no longer constructs a TM to embody a theory (except for didactic reasons or to study questions about foundations). Rather one writes *programs* to make theoretical predictions. But the primitives of the theory come from thinking about real phenomena, for the efficient discussion of which theory is invented. Theory, after all, is essentially a language (or collection of languages) for discussing the world of observation and experiment. Improved theories are becoming more and more synonymous with better programs or computers for that purpose. Computer paradigms for biology are conceptually and philosophically neutral in the sense that *any* logical theory now comes under that paradigm.

4. IRREVERSIBILITY AND QUANTUM MEASUREMENT

The operational arrow of time is pointed by the *possibility* of acquiring information in the future. As dynamics (classical or quantum) is invariant under time reversal the arrow direction is undecidable in those theories! In the sequel we show that quantum characterization of the measurement process shows that quantum mechanics is indeed incomplete in both Goedel's and Einstein's senses (here identical). Denial of incompleteness entails *inconsistency*: we can construct mutually exclusive quantal descriptions of the same physical situation with equal operational legitimacy. The *macroscopic* requirement of *irreversibility* of measurement (uses of mixtures rather than wave functions to express results of measurement, preparation, or information acquisition) saves the theory from formal inconsistency.

The argument resembles the Einstein–Podolsky–Rosen paradox, Einstein (1935), but goes beyond it in two respects. First, measurements are made (of complementary observables) on *both* of the separated partial systems, and second, the macroscopically recorded results are observed by an *ensemble* of macroscopic, mutually noninterfering, Lorentz-equivalent observers. These fall into equivalence classes, in the sense that all members of the same class agree on the appropriate quantum descriptions of the results, but members of different classes have inconsistent descriptions until the wave functions they use are replaced by mixtures. In short, the quantum descriptions, though complete, are not objective, i.e., inconsistent, while the objective descriptions are statistical mixtures and so are not complete. The mathematical argument is elementary.

As in EPR, let a compound system S_{12} with wave function ψ_{12} split into spatially separated subsystems S_1, S_2 with respective wave functions ψ_1, ψ_2. Let $\{u_i^1\}$ and $\{u_i^2\}$ be complete sets of eigenfunctions of observable U for S_1 and S_2, respectively, likewise $\{v_i^1\}$ and $\{v_i^2\}$ for the *incompatible* observable V. We can expand ψ_{12} in two different ways, namely,

$$\psi_{12} = \sum_i c_i^U u_i^1 u_i^2 \tag{31}$$

$$\psi_{12} = \sum_i c_i^V v_i^1 v_i^2 \tag{32}$$

where the c_i^U and c_i^V are expansion coefficients.

As in EPR, measuring U on S_1 gives (suppressing the now superfluous subscript i)

$$\psi_1 = u^1 \tag{33}$$

$$\psi_{12} = u^1 u^2 \tag{34}$$

whence the wave function of S_1, measured to be u^1 directly, requires the inference that the wave function of S_2 be u^2, which we write

$$\psi_2(1) = u^2 \tag{35}$$

indicating explicitly that this function for ψ_2 is conditioned on a measurement made on S_1. Similarly, if we measure V on S_2 we get

$$\psi_2 = v^2 \tag{36}$$

$$\psi_{12} = v^1 v^2 \tag{37}$$

$$\psi_1(2) = v^1 \tag{38}$$

If both measurements are carried out independently on the separated systems with a *spacelike interval between the two measurement events*, then the totality of Lorentz-equivalent observers splits up into two large classes and a singular (small) class as follows. One large class, L_1 say, notes the result of measuring U on S_1 before being able to note the result of measuring V on S_2. A second large class, L_2, sees the result of measuring V on S_2 before the result of measuring U on S_1, while the singular class L_0 (the only nonrelativistic one) sees the results of both measurements simultaneously.

Observers in L_0 receive apparently equally valid, but mutually contradictory, pieces of information about each system, namely, that the state of S_1 is both u^1 (measured directly) and v^1 (inferred indirectly from the directly measured state of S_2), while S_2 is assigned state v^2 (directly) and u^2 (indirectly). But since U and V were chosen as complementary observables, neither S_1 nor S_2, according to quantum mechanics, can simultaneously be in the pair of eigenstates assigned to it. We have two mutually exclusive descriptions of the state of each system. One might argue that the double intervention so decouples S_1 and S_2, that the measurements have prepared S_1 in state u_1, S_2 in state v_2, and that the inferred states are thus incorrectly inferred. But how should S_1 and S_2 "know" this is wrong if they are widely separated? One might dismiss this as ignorance of the answer to EPR given in Bohr (1935), but it is harder to dismiss the following. Because S_1 and S_2 have become independent systems, U measured on one commutes with V measured on the other. The eigenfunctions of this $U \otimes V$ or $V \otimes U$ operator are $x^i y^j$, where the x's and y's can be freely chosen as u's or v's, while i and j are 1 or 2. But there can be no quantum observable $U \otimes V$ for S_{12} such that

$$\psi_{12} = \sum_i c_i^{U \otimes V} u_i^1 v_i^2 \tag{39}$$

or

$$\psi_{12} = \sum_i c_i^{V \otimes U} v_i^1 u_i^2 \tag{40}$$

Einstein can now laugh and say that QM is incomplete because only half of the logically possible and operationally permissible distinct experimental situations find their place in quantum descriptions of S_{12}, though all are admitted in the theory of $S_1 \otimes S_2$.

We believe the only way to respond to this criticism which does not dodge or befuddle the issue is to admit its truth and to understand what it means. The change from S_{12} to $S_1 \otimes S_2$ is a real change in the specification of the system, equivalent to putting an impenetrable wall between S_1 and S_2, as in Lamb (1969). There is an entropy or free energy price to pay for this macroscopic act of preparing the system anew as discussed earlier, and simply illustrated by the following thought experiment.

Consider a container of volume $2V$ with a slidable partition P capable of dividing it into two mutually inaccessible regions, each of volume V. With the partition out introduce two systems A and B. They can be thought of as molecules in the spirit of Maxwell and Gibbs. Because they are free to interact (e.g., collide in billiard-ball fashion), we can consider them to be a compound system AB. Now introduce partition P. There will be four macroscopically distinguishable, mutually exclusive and exhaustive situations which we designate as PAB, APB, BPA, and ABP. They respectively represent both A and B to the right of P, A to the left and B to the right of P, B to the left and A to the right of P, and both A and B to the left of P. As in Szilard (1929), acquisition of information as to which of the four situations actually results entails an entropy cost ΔS of $k \ln 4$ if the four are viewed as equally probable a priori. If A and B are identical molecules, the Gibbs paradox forces identification of APB and BPA, reducing ΔS to $k \ln 3$. If A and B are so strongly attracted to each other that there is negligible probability of their ending up on opposite sides of P, or if their mutual repulsion is so great that there is negligible probability of finding them on the same side of P, ΔS becomes $k \ln 2$. In any case, as long as there is no reason to say A and B will prefer the left or right volume V, and even if the energy ΔE needed to slide P in is zero, the entropy cost of making the distinctions contemplated is in the range of one or two times $k \ln 2$.

We can translate the thought experiment into a description of the earlier discussion of the quantum situation. The systems S_1 and S_2 replace A and B, with S_{12} corresponding to the partial systems in direct interaction, i.e., constituting a single system before separation (volume $2V$), while the introduction of P represents the fourfold splitting of S_{12} into the separated system $S_1 \otimes S_2$ and the choice of U or V for measurement on each one (left or right). Then ABP corresponds to the usual EPR situation in which U is measured on one (or both) subsystems, and PAB to the analogous case for V. The cases APB and BPA, belonging to no quantum observable meaningful on S_{12}, correspond, respectively, to measuring U on S_1 and V on S_2 or V on S_1 and U on S_2. They are certainly operationally and macroscopically meaningful nevertheless. So Einstein's charge of incompleteness for quantum mechanics seems irrefutable in a real sense, but the incompleteness is much like the puzzle of irreversibility faced by Boltzmann or Gibbs in the classical case. In both the classical and quantal situations the fundamental dynamics is reversible, so *a new consideration must be introduced*, of necessity *not derivable from the dynamics, to explain irreversibility*. Historically this has been either an assumption of molecular chaos or an appeal to coarse-graining and the phase-space streaming entailed by Lionville's theorem. The latter, as shown in Rothstein (1957, 1974), corresponds neatly to the operational situation, for any operationally defined measurement or preparation scheme corresponds to splitting phase space up into cells of measure corresponding to the operationally distinguishable alternatives involved. Phase space streaming then spreads the ensemble of representative points of any one cell into a filament threading many cells. Now distinctions between points of any one cell are operationally undefined within the specification by which the cells are defined. The dynamical evolution of the ensemble corresponding to maximal information or most precisely controlled preparation (one cell) thus requires monotonic increase in the number of cells needed to cover the set of streamed representative points. As the entropy is $k \log \phi$ where $\phi = N\Delta\tau$, with $\Delta\tau$ the phase volume of a cell, and N the number of cells threaded by the filament, we get monotonic entropy increase. But note the *change* in what corresponds to a system in going from the microscopic to the macroscopic (i.e., operational) situations. *The representative point* (representing a microscopic system) *is suddenly replaced by an ensemble of points* (representing a macroscopic system). Ensembles spread, points do not. The irreversibility results from the transition from a (hidden) microscopic situation to the operational macroscopic situation, which demands an ensemble description. Similarly the entropy increase associated with any contemplated measurement scheme or preparation act in QM stems from a macroscopic source. It implies mixtures (and loss of some phase information) rather than pure states to describe the results. The incompleteness of description only implicit in the classical case becomes explicit in the quantum case. It is a virtue of QM, rather than a shortcoming, that this is so. We believe that any successor microscopic theory to current quantum mechanics will of necessity display similar incompleteness when measurement or irreversibility is analyzed operationally.

We discussed the class L_0 of equivalent observers in detail, despite its singular nature, for several reasons. First, it is the only case that comes up nonrelativistically. As Schrödinger's equation is nonrelativistic this case had to be examined. Second, it permitted discussion of the problems of completeness, irreversibility (both of measurement and in general), and their connection with ensembles and the change in viewpoint on going from microscopic or macroscopic. Third, this could be done without mixing in questions of subjectivity, objectivity, consistency, and causality. These are involved for the classes L_1 and L_2, which we take up next.

Historically, questions about the completeness of QM were motivated by a requirement that fundamental theory describe what is physically "real." The Einstein–Bohr debates (Schilpp, 1949) showed how subtle and elusive that concept could be. But surely all can accept that what competent observers agree on (i.e., what can be therefore called objective) stands a much better chance of being "real" than what competent observers disagree about (which we can label subjective). *If two conflicting statements are equally well grounded in fundamental theory, the theory is inconsistent unless the question as to which one is correct is undecidable in principle, i.e., the theory is incomplete.* As Goedel showed that *no* theory complex enough to include arithmetic could be proved, within the system, to be both consistent and complete, it is not to be counted as defect of QM if such a case is shown to exist in it. We now exhibit one.

The class of observers L_1 records results of the following procedure. Measurement of U on S_1 is followed by measurement of (incompatible) V on S_2, where the measurement events are separated by a spacelike interval. By (31), (33), (34), and (35) this means S_1 is found to be in state u^1, forcing S_2 to be in u^2 until measurement of V puts it in state v^2. Similarly the class L_2 observes a measurement of V on S_2 followed by measurement of U on S_1, the two events being separated by a spacelike interval (interval signature is preserved under Lorentz transformation between L_1 and L_2). By (32), (36), (37), and (38) this means S_2 is found in state v^2, forcing S_1 to be in state v^1 until measurement of U puts it into u^1. So L_1 and L_2, who are equally competent if one accepts special relativity, disagree about what the states of S_1 and S_2 are during the interval between measurements. *Both* views, after one measurement, are therefore subjective, i.e., inconsistent with each other. They agree on the results produced by carrying out both procedures, which therefore are objective, though not consistent with any eigenfunction description of S_{12} unless it be a *mixture*.

The foregoing shows that, if measurement can prepare a pure state, the same operational situation can be described by mutually exclusive subjective wave functions, by equally competent observers. The assumption that a pure state is a *complete* description (of an objective "reality") thus leads to contradiction (formal inconsistency). The objective description of that situation, on the other hand, is consistent, but necessarily incomplete. The QM description of S_{12} can only generate mixtures as objective states of $S_1 \otimes S_2$, or, what is the same thing, irreversibility inheres in the acquisition of objective information about, or the preparation of, a real system. In a sense, *this derives the second law of thermodynamics, via relativity, as a necessary condition for the operational consistency of quantum mechanics*. This in turn, as shown in Rothstein (1957, 1964), leads to decay of relevance of information about the state of a closed system for predicting or retrodicting its state in the remote future or past, respectively. The process is formally identical to approach to equilibrium; information decrease is proportional to entropy increase. It can also be equated to a *weakening of causal connection* in the sense that correlations between a state as measured or prepared at some time t_0 and states increasingly remote in time from it approach zero. Also, time does have an arrow, corresponding to the actual entropy increase accompanying measurement or preparation. This differs *physically* from the *conceptual* increase in fuzziness associated with retrodiction even though the formalism is symmetrical, for the latter admits no *new* possibilities for measurement or preparation acts on past situations.

Turning now to computation and biology, we see that the coupled chains of selective acts (many chains can go on in parallel) which constitute their characteristic behavior *inherently* demand cooccurring dissipation of free energy. Although the rate of energy dissipation in computers can be enormously reduced by miniaturization or low-temperature operation, there is an irreducible entropy increase associated with each selective act. This inheres in the operational situation for all physical SSs. It is *not* clear that a minimum energy price, independent of temperature, for imposing or relaxing a constraint need exist, from a fundamental point of view.

Though computers and living things are macroscopic systems, they (and other SSs) differ in a fundamental sense from what physicists normally mean by the term. A general purpose computer without a program does not compute. It is inert. It becomes a *specific* computer when given a program, and the machine *ordinarily* behaves differently with different programs (whether two programs or algorithms are equivalent is *not* decidable in general, however). So a computer (or conditionable, teachable, adaptable living thing) can sometimes be conveniently viewed as an incompletely specified machine, as an ensemble of possible machines, or as a single well-specified machine responding to a well-defined class of inputs containing both program and data to be processed in accordance with it. In living things a *microscopic* event can alter a genetic program, say, making the engineer's or physicist's way of specifying the system difficult to realize, to say the least. The practice of the biologist to secure homogeneous populations by selective breeding, cloning, or the like, may long be the only practical way to "prepare," or initiate a complex preparation procedure, serving to define a particular kind of living SS operationally.

Progress in theoretical biology, we believe, needs help from more than physics and biology. It needs help from computer science, where SSs can be studied without quantum complications, and where any behavior which can be described by either a statistical, mathematical, or logical theory can be modeled. In particular, reconfigurable systems consisting of interacting subsystems should be studied intensively. We do not deny the importance of quantum mechanics for biology, but believe too exclusive a preoccupation with it would tend to neglect some crucial things. They are first that SS behavior is only "piecewise quantum mechanical" in each sufficiently small subsystem in the intervals between irreversible actions. At the grosser level of description in terms of such actions ("bit level") computerlike models seem very appropriate. At this level much of the thermodynamic-macroscopic-engineering way of dealing with systems seems proper, augmented by the computer-scientist's approach. The second crucial area is where the microscopic meets the macroscopic, where nonequilibrium thermodynamics meets the time varying Schrödinger equation, where measurement, information, irreversibility, control, and metastability meet computation and system reconfiguration. We believe many future battles of theoretical biology will be fought here, along with many highly relevant skirmishes in the area of fundamental limits on computation.

5. CONCLUDING DISCUSSION

A number of aspects of the statistical thermodynamics of physical systems capable of exhibiting selective behavior have been studied. The class of such systems includes computers and measuring apparatus, and to the extent it is justified to consider living systems as being physical systems, includes them as well.

Some of the chief results are the following. There is an irreducible entropy increase associated with every selective act, which depends on the selectivity of the act and thus applies to any system capable of doing it. This constant entropy increase persists even as absolute zero is approached (this may well lead, therefore, to a generalized form of the third law of thermodynamics). However, there is no evidence for a temperature-independent minimum energy cost, characteristic of the act, but there is the usual quantum limit characteristic of the particular system performing the act.

Since measurement (or preparation) was shown to be both the essence of selective behavior, and the source of irreversibility on phenomenological grounds, it became necessary to examine both computation and measurement per se from as fundamental a view as possible. This meant both the logical and quantum levels, the first being foundations of computability, formal systems, automata, Turing machines and Goedel's theorems, the second being an examination of how *measurement* linked up with the *theoretical* structure of QM. The result was that irreversibility of measurement was needed to prevent QM from asserting a contradiction in the form of mutually exclusive, equally justified assertions about the state of a system. The connection with the historic debate between Einstein and Bohr was exhibited and hereby declared a draw. Goedel's theorems show that true theorems can exist in a formal system which are not provable in the system (i.e., the system need not be complete) and that the consistency of a system (i.e., proof within the system that no contradiction could ever be derived) was such a theorem. Analysis of subjectivity, objectivity, completeness, and consistency for quantum measurement then showed, in effect, that implementation of Einstein's requirement of completeness for QM would make it inconsistent, but that his notion of physical reality might be true but capable neither of proof nor disproof. Now one can always augment the axioms of a consistent but incomplete formal system with a new one, consisting of a theorem which is true but not provable in the system, to obtain a new consistent system. All true theorems of the original system remain true in the new one, but many new true theorems now become accessible by proof. So Einstein's program of trying to find a theory from which QM can be derived, but which would go beyond it in directions where QM must be mute, is both a logical possibility and an intellectual challenge.

The computer paradigm for biological theory is philosophically neutral in the following sense. Any theory deserving the appellation "theory" can be logically modeled by a suitable Turing machine. Furthermore, one can construct a big TM from others, which become its submachines. Now as biological mechanisms become better understood, they become amenable to theoretical formulation and thus to computer simulation. If all the mechanisms were understood the corresponding programs could all be put together to form a large program of which the original programs are now subroutines. The corresponding Turing machine would be a computer model for biology. The method, being applicable to any theory, does not bias the theory philosophically. Computer models are ways of talking about real systems of interest in computationally, mathematically, and logically efficient ways, with tremendous data storage, retrieval, and reduction capabilities besides. Specific computer models will become outmoded and discarded, but computer models as a class will never become outmoded as long as theory, mathematics, and logic retain their meanings.

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